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Rocket Powered Flight as a Perturbation to the Two-Body Problem.

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by

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ABSTRACT

Rocket Powered Flight as a Perturbation to the Two–Body Problem

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The two body problem and the rocket equation $\ddot{\mathbf{r}}+\epsilon\,\alpha\,\dot{\mathbf{r}}+\frac{k}{r^3}\,\mathbf{r}=\mathbf{0}$ have been expressed in numerous ways. However, the combination of the rocket equation with the two body problem has not been studied to any degree of depth due to the intractability of the resulting non-linear, non-homogeneous equations. The goal is to use perturbation techniques to approximate solutions to the combined two body and rocket equations.

DEDICATION

I dedicate this thesis to Tara, my wife who shares my love.

ACKNOWLEDGMENTS

I especially thank Dr. Jeff Knisley for spending many patient hours in guiding me. This thesis would never have happened without you. I also thank Dr. Lyndell Kerley whose classes helped prepare me for this moment. I really, really thank Dr. Robert Gardner for giving me help when I needed it. Thanks Dr. Bob! I also thank my parents, Roy and Nancy Clark, for their support. Most important, I thank God, without His help I wouldn't have gotten this far.

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1 THE TWO BODY PROBLEM

In this chapter we will discuss the background of the rocket equation. We will also discuss solutions to the two body problem. We will review a previous problem and see how perturbation techniques can be applied to the two body problem with a rocket.

1.1 Background

The two body problem was first solved by Sir Isaac Newton, and since then, it has been expressed in a number of different ways. Similarly, the rocket equation was first developed in 1903 by Konstantin Tsiolkovsky to explain the operation of a rocket with the absence of gravity. However, the combination of the rocket equation with the two body problem has not been studied to any degree of depth due to the intractability of the resulting non-linear, non-homogeneous equations. In this project, we will use perturbation techniques to approximate solutions to the combined two body and rocket equations.

1.2 Kepler's Problem

We first discuss results leading to the work on the two-body problem. These results were used in the simulations created during Spring 2004. These formulas can be found in the classical celestial mechanics text written by Harry Pollard. [3]

First, let us consider the behavior of a particle attracted to a fixed center. In this thesis, if $\mathbf{u}$ is a vector, then $u = \|\mathbf{u}\|$. Also, the following definitions will be important.

Definition 1.1 *If a particle has a position* $\mathbf{r}(t)$ *at time* t*, then its angular velocity is*

$$\mathbf{L} = \mathbf{r} \times \mathbf{v}$$

where $\mathbf{v} = \dot{\mathbf{r}}$ *is the velocity, which is the time derivative, of* $\mathbf{r}(t)$*.*

Definition 1.2 *A particle is said to be* ***attracted to a fixed center*** *if its position* $\mathbf{r}(t)$ *at time* t *satisfies a differential equation of the form*

$$\ddot{\mathbf{r}} = -f(\mathbf{r})\frac{\mathbf{r}}{r}$$

where $f(\mathbf{r})$ *is positive and differentiable except possibly at the origin.*

Proposition 1.3 *If a particle with position* $\mathbf{r}(t)$ *is attracted to a fixed center, then its angular velocity* $\mathbf{L}$ *is constant.*

Proof: Differentiating $\mathbf{L}$ with respect to t yields

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mathbf{r} \times \frac{d\mathbf{v}}{dt} \\ &= \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \left(-f(\mathbf{r})\frac{\mathbf{r}}{r}\right) \\ &= \mathbf{0}\,. \qquad \square \end{aligned}$$

Since $\mathbf{L}$ is constant and is perpendicular to $\mathbf{v}(t)$ for all times t for which it is defined, the motion of a particle attracted to a fixed center is necessarily in a plane. We assume that the plane is the xy-plane and that the x-axis corresponds to the position of the particle at time $t = 0$ (as long as it is not the origin). Also, if we define

$$\mathbf{u} = \langle \cos(\theta), \sin(\theta) \rangle\,, \qquad \mathbf{u}' = \langle -\sin(\theta), \cos(\theta) \rangle$$

where θ is the polar angle, then $\mathbf{r} = r\mathbf{u}$ and

$$\mathbf{v} = \frac{dr}{dt}\mathbf{u} + r\mathbf{u}'\frac{d\theta}{dt} \ .$$

It follows that

$$\mathbf{r} \times \mathbf{u} = r^2\frac{d\theta}{dt}\left(\mathbf{u} \times \mathbf{u}'\right) \ .$$

If $A(t)$ denotes the area swept out by the position vector from time 0 to time t, then in polar coordinates

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{L}{2} \ .$$

Since L is constant, the position vector of a particle attracted to a fixed center sweeps out equal areas in equal times (Kepler's second law).

The next 2 theorems combined are known as Sundman's theorem of total collapse.

Theorem 1.4 *If a particle subject to attraction by a fixed center starts from rest(i.e.,* $\mathbf{v} = \mathbf{0}$ *at some instant* $t = 0$*), then the motion is along a line through the origin.*

Proof: Since $\mathbf{v} = \mathbf{0}$ at time $t = 0$, the angular velocity $\mathbf{L} = \mathbf{0}$ at time $t = 0$, and since $\mathbf{L}$ is constant, it is zero for all times where $\mathbf{r}(t)$ is defined. Note that

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{(\mathbf{r} \times \mathbf{v}) \times \mathbf{r}}{r^3} = \frac{\mathbf{L} \times \mathbf{r}}{r^3} \ .$$

Then for all times t where the position is defined, we have

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{\mathbf{0} \times \mathbf{r}}{r^3} = \mathbf{0} \ .$$

This means $\frac{\mathbf{r}}{r}$ remains constant. Thus, the particle must be traveling toward the center of the force along a straight line. $\square$

In this chapter, the forcing function is the inverse square law

$$f(\mathbf{r}) = \frac{k}{r^2} \; .$$

Notice that $f(\mathbf{r}) > 0$ for all $\mathbf{r} \neq 0$.

Theorem 1.5 *If a particle starts from rest in an inverse square field, then it must collide with the center of force in a finite length of time.*

Proof: The acceleration due to gravity acting upon the rocket is given as

$$\ddot{\mathbf{r}} = -\frac{k}{r^2}\mathbf{r} \tag{1}$$

where k is a positive constant depending only on the units chosen and the source of attraction. So,

$$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = -\frac{k}{r^2}\mathbf{r} \cdot \dot{\mathbf{r}} \tag{2}$$

and

$$v^2 - \frac{k}{r} = h \tag{3}$$

where h is constant. This implies

$$v^2 = h + \frac{k}{r} \to \infty \tag{4}$$

as r approaches 0. Therefore, it is only a finite amount of time t before the object reaches the center. $\square$

As we look at the behavior of particles, we can also look at the relationship between the position $\mathbf{r}$, velocity $\mathbf{v}$ and angular momentum $\mathbf{L}$.

Theorem 1.6 *The velocity, position, and angular momentum are related by the equation*

$$v^2 = \dot{r}^2 + L^2 r^{-2} \ .$$

Proof: Let $\mathbf{a} = \mathbf{r}$, $\mathbf{b} = \mathbf{v}$ in the vector formula

$$(\mathbf{a} \cdot \mathbf{b})^2 + \|\mathbf{a} \times \mathbf{b}\|^2 = a^2 b^2 \ . \tag{5}$$

Where the standard vector formulas are

$$\|\mathbf{a} \times \mathbf{b}\| = ab \sin(\theta)$$

and

$$\mathbf{a} \cdot \mathbf{b} = ab \cos(\theta) \ .$$

That is we can write equation (5) as

$$(\mathbf{r} \cdot \mathbf{v})^2 + \|\mathbf{r} \times \mathbf{v}\|^2 = r^2 v^2 \ .$$

Since $L = \|\mathbf{r} \times \mathbf{v}\|$, we have

$$(\mathbf{r} \cdot \mathbf{v})^2 + L^2 = r^2 v^2 \ .$$

Dividing both sides by r^2 yields

$$v^2 = (\mathbf{r} \cdot \mathbf{v})^2 r^{-2} + L^2 r^{-2} \ .$$

Since $r = (\mathbf{r} \cdot \mathbf{r})^{\frac{1}{2}}$, we have

$$\begin{aligned}
\dot{r} &= \frac{1}{2} (\mathbf{r} \cdot \mathbf{r})^{-\frac{1}{2}} (\mathbf{v} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{v}) \\
&= (\mathbf{r} \cdot \mathbf{r})^{-\frac{1}{2}} (\mathbf{r} \cdot \mathbf{v}) \\
&= r^{-1} (\mathbf{r} \cdot \mathbf{v}) \ .
\end{aligned}$$

Hence,

$$\dot{r}^2 = r^{-2} \left(\mathbf{r} \cdot \mathbf{v}\right)^2 \ .$$

Recall that

$$v^2 = \left(\mathbf{r} \cdot \mathbf{v}\right)^2 r^{-2} + L^2 r^{-2} \ .$$

Thus we may conclude that

$$v^2 = \dot{r}^2 + L^2 r^{-2} \ . \qquad \square$$

Kepler's problem is the differential equation

$$\ddot{\mathbf{r}} = \frac{-k}{r^3}\mathbf{r} \ .$$

In order to show that solutions to Kepler's problem are conic sections, we may use the Lenz vector.

Definition 1.7 *The **Lenz Vector** for the Kepler problem is defined as*

$$\mathbf{F} = -\frac{k}{r}\mathbf{r} + \mathbf{v} \times \mathbf{L} \ .$$

Theorem 1.8 *The Lenz vector,* $\mathbf{F}$*, is constant.*

Proof: The definition of the Lenz vector implies that

$$\begin{aligned} \frac{d\mathbf{F}}{dt} &= \frac{k}{2r^3}\left(\frac{d}{dt}\mathbf{r} \cdot \mathbf{r}\right)\mathbf{r} - \frac{k}{r}\mathbf{v} + \dot{\mathbf{v}} \times \mathbf{L} \\ &= \frac{k}{r^3}\left[\left(\mathbf{r} \cdot \mathbf{v}\right)\mathbf{r} - \left(\mathbf{r} \cdot \mathbf{r}\right)\mathbf{v}\right] + \left(\frac{-k}{r^3}\ \mathbf{r}\right) \times \mathbf{L} \\ &= \frac{k}{r^3}\left[\mathbf{r} \times \left(\mathbf{r} \times \mathbf{v}\right)\right] - \frac{k}{r^3}\left[\mathbf{r} \times \left(\mathbf{r} \times \mathbf{v}\right)\right] \\ &= \mathbf{0} \ . \qquad \square \end{aligned}$$

Theorem 1.9 ***(Kepler's First Law)*** *Solutions to Kepler's problem are conics.*

Proof: By taking the dot product of the Lenz vector and the position vector, we obtain

$$\mathbf{F} \cdot \mathbf{r} = \frac{-k}{r} \, \mathbf{r} \cdot \mathbf{r} + (\mathbf{v} \times \mathbf{L}) \cdot \mathbf{r} \ .$$

Which is

$$rc\cos(\theta) = -kr + (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{L} \ .$$

Where $c = \|\mathbf{F}\|$ is constant. So

$$r\,[k + c\cos(\theta)] = L^2 \ .$$

Which implies

$$\begin{aligned} r &= \frac{L^2}{k + c\cos(\theta)} \\ &= \frac{L^2/k}{1 + c/k\cos(\theta)} \\ &= \frac{L^2/k}{1 + e\cos(\theta)} \ . \end{aligned}$$

Where e is the eccentricity of the conic. $\square$

Lets look more closely at the orbit of the rocket by showing that it can be described as a conic.

Theorem 1.10 *If* $0 < e < 1$ *or* $e > 1$ *the semi-major axis of the corresponding conic has length* a *given by the formula*

$$ka|e^2 - 1| = L^2 \ .$$

Where e *is the eccentricity of the orbit and* k *is a constant defined such that*

$$k = MG \ .$$

Proof: We begin with the formula for an conic,

$$r = \frac{L^2/k}{1 + e\cos(\theta)} \ .$$

At the closest point we have

$$r(0) = \frac{L^2/k}{1 + e\cos(0)}$$

so

$$r(0) = \frac{L^2/k}{1 + e} \ .$$

At the furthest point we have

$$r(\pi) = \frac{L^2/k}{1 + e\cos(\pi)}$$

so

$$r(\pi) = \frac{L^2/k}{1 - e} \ .$$

If we add the two lengths together, we get

$$r(0) + r(\pi) = 2a$$

then

$$\frac{L^2/k}{1 + e} + \frac{L^2/k}{1 - e} = 2a \ .$$

Multiplying both sides of the equation by the identity

$$(1 + e)(1 - e) = 1 - e^2$$

we have

$$\frac{2L^2}{k} = 2a(1 - e^2) \ .$$

Thus,

$$L^2 = ak(1 - e^2) \ .$$

Therefore, for $0 < e < 1$ and $e > 1$ the equation becomes

$$L^2 = ak|1 - e^2|$$

with a the length of the semi-major axis of the corresponding conic. $\square$

Upon consideration of the magnitude of the angular momentum of the rocket (L), we consider the orbit of the particle. The following shows that Kepler's Third Law is a corollary to Kepler's Second Law, given that the period p of a particle is the time it takes to sweep out the area once.

Theorem 1.11 ***(Kepler's Third Law)*** *If* $0 < e < 1$, *then*

$$p = \left(\frac{2\pi}{\sqrt{k}}\right) a^{\frac{3}{2}}$$

Proof: The total area (A) of an ellipse is

$$A = \pi a^2 (1 - e^2)^{1/2}$$

and we know that

$$L \neq 0 \Rightarrow \frac{dA}{dt} = \frac{L}{2} \ .$$

Then at time t we have

$$A = \frac{L}{2}\, t \ .$$

So at $t = p$

$$A = \frac{L}{2}\, p \ .$$

Hence at time $t = p$ we have the equality

$$\pi a^2(1-e^2)^{\frac{1}{2}} = \frac{L}{2}\ p$$

then

$$\frac{2}{L}\ \pi a^2(1-e^2)^{\frac{1}{2}} = p\ .$$

Note that from an earlier result

$$L^2 = ak|1-e^2|$$

and since $0 < e < 1$ we have

$$L^2 = ak(1-e^2)$$

which implies

$$1 - e^2 = \frac{L^2}{ak}\ .$$

By substitution we now have

$$\frac{2}{L\pi a^2}\left(\frac{L^2}{ak}\right)^{\frac{1}{2}} = p\ .$$

So

$$\frac{2}{L}\ \pi a^2 \frac{L}{a^{\frac{1}{2}}k^{\frac{1}{2}}} = p\ .$$

Thus

$$\frac{2\pi a^{\frac{3}{2}}}{k^{\frac{1}{2}}} = p\ .$$

Therefore we have

$$\left(\frac{2\pi}{\sqrt{k}}\right) a^{\frac{3}{2}} = p$$

the period of the particle. $\square$

1.3 Simulation with a Maple Program

A Maple program was created for the implementation of these results. The program involves a regularizing transformation.

We first began by defining the rocket's initial position $\mathbf{r_0}$ (e.g., 7,000,000 meters) and velocity $\mathbf{v_0}$ (e.g., 10,000 meters per second). Given the mass of Earth,

$$M = 5.976 \times 10^{24} \text{ kg}$$

and the universal gravitational constant

$$G = 6.672 \times 10^{-11} \ \frac{\text{m}^3}{\text{kg s}^2}$$

we have the effect due to gravity (k) to be constant where

$$k = MG \ .$$

The angular momentum (L) is defined as

$$L = \|\mathbf{r_0} \times \mathbf{v_0}\|$$

and the Hamiltonian energy (h) is

$$h = \frac{1}{2} \ \|\mathbf{v_0}\|^2 - \frac{k}{\|\mathbf{r_0}\|} \ .$$

A long but elementary calculation shows us that the eccentricity of the orbit (e) is

$$e = \sqrt{1 + 2 \ h \ \frac{L^2}{k^2}} \ .$$

Suppose that $0 < e < 1$. Then the circle centered at $(-ae, 0)$ with radius a circumscribes the elliptical orbit and defines an angle E called the **eccentric anomaly**. (See figure 1).

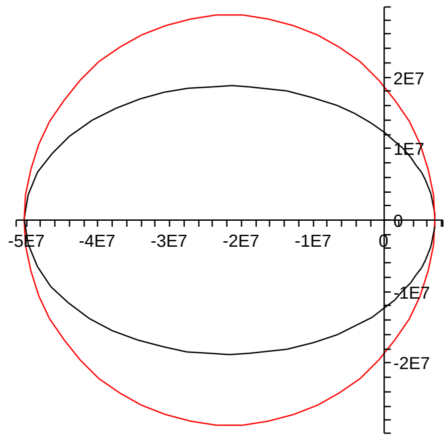

Figure 1: Orbit of Particle Contained within Regularized Orbit.

To determine the position of the particle on the elliptical orbit, we define a new independent variable u by

$$u = k \int_0^t \frac{d\tau}{r(\tau)}$$

called a **regularizing variable**. In terms of the regularizing variable u, the time, polar distance, and polar angle are given by

$$\begin{aligned} t &= \frac{a^{3/2}\,(u - e\sin(u))}{\sqrt{k}} \\ r &= a\,(1 - e\cos(u)) \\ \theta &= 2\ \arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{u}{2}\right)\right) \ . \end{aligned}$$

To show the position of the particle on its orbit at time t, we incorporated the regularizing variable into a FOR loop. This demonstrates that at any time (t) we can show the rockets exact position in it's elliptical orbit by tracing along the regularized path.

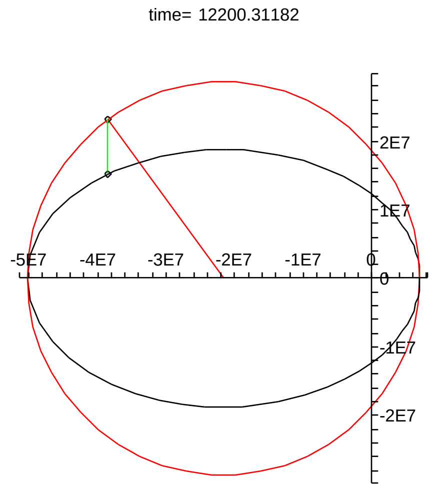

Figure 2: Particle's Position in Orbit Compared with Regularized Orbit.

The Maple code for this project can be found in appendix 1.

2 PREVIOUS RESEARCH

In this chapter we will view research that has been done on the subject of two–body and rocket problems. In particular, we observe the effects of continuous thrust (Battin [1]). What we attempt in this thesis is the exact opposite. We wish to minimize the thrust and essentially have no use of fuel. Allowing the rocket to be propelled by its own momentum.

2.1 Rocket Powered Flight in an Inverse Square Field

By beginning with the mass of the rocket (M_R), along with the velocity of the rocket ($\dot{\mathbf{r}}$), we have the linear momentum of the rocket $M_R\ddot{\mathbf{r}}$. In this result, we let $\boldsymbol{\mu}$ equal the velocity of the exhaust gas, then the differential of the exhaust momentum can be written as $dM_R\boldsymbol{\mu}$ (Smith [4]).

By accounting for the acceleration due to gravity, we have

$$\mathbf{a_c} = \frac{k}{r^3}\mathbf{r}$$

where k is the universal gravitational constant and $\mathbf{a_c}$ is the acceleration due to the effect of gravity.

By Newton's third law, we have conservation of linear momentum. This implies that

$$\underbrace{(M_R\ddot{\mathbf{r}})\,dt}_{rocket} + \underbrace{d(M_R)\boldsymbol{\mu}}_{exhaust} + \underbrace{\left(M_R\frac{k}{r^3}\mathbf{r}\right)dt}_{gravity} = \mathbf{0}$$

which aids us in developing a formulation for rocket powered flight.

To use the above equation for conservation of momentum, we first define $\boldsymbol{\mu}$. We assume that $\boldsymbol{\mu}$ is parallel to $\mathbf{v}$, which means that there is a scalar function $\alpha(t)$ for which

$$\boldsymbol{\mu} = \alpha(t)\dot{\mathbf{r}} \ .$$

By substituting for $\boldsymbol{\mu}$ into the conservation of momentum equation, the result is

$$\ddot{\mathbf{r}} + \epsilon\alpha\dot{\mathbf{r}} + \frac{k}{r^3}\mathbf{r} = \mathbf{0}$$

where $\epsilon = \frac{d(M_R)}{M_R} = \frac{\dot{m}}{m}$ is the percentage of fuel loss. This is the version of the rocket equation we will be using.

2.2 Continuous Thrust

Suppose there is a rocket in orbit. This rocket is initially in a circular orbit of radius r and at time $t = t_o$ there is a continuous thrust. This allows the tangential acceleration a_t to be constant. This makes the velocity of the rocket $\mathbf{v}$ in a direction tangent to the orbit, *i.e.*, $\mathbf{v} = v\mathbf{T}$, where $\mathbf{T}$ is the direction of the force applied tangent to the orbit (Battin [1]).

Hence the acceleration is

$$\mathbf{a} = \frac{dv}{dt}\mathbf{T} + v\frac{d\mathbf{T}}{dt} \tag{6}$$

where

$$\left\|\frac{d\mathbf{T}}{ds}\right\| = \kappa$$

where s is the arc length of the orbit and κ is the curvature of the orbit defined as

$\kappa = \frac{1}{\rho}$, where ρ is the radius of curvature. So we have,

$$\left\| \frac{d\mathbf{T}}{dt} \right\| = \left\| \frac{d\mathbf{T}}{ds} \frac{ds}{dt} \right\| = \kappa v \ .$$

This makes equation (6) become

$$\mathbf{a} = \frac{dv}{dt}\mathbf{T} + \kappa v^2 \mathbf{N}$$

where $\mathbf{N}$ is the direction of the force applied normal to the rocket's position in orbit. And so, we express the total acceleration of the rocket $\mathbf{a}$ as

$$\mathbf{a} = a_t \mathbf{T} \ \underbrace{-\frac{\mu}{r^2}\mathbf{U}}_{energy\ loss} \tag{7}$$

where μ is a constant related to the thrust of the rocket's engine dependent on the rate of fuel loss and $\mathbf{U}$ is the direction of the force applied toward the center along the radius of the rocket's position in orbit, *i.e.*, $\mathbf{U} = \frac{\mathbf{r}}{r}$.

In terms of components in the tangential and normal directions, we have,

$$-\frac{\mu}{r^2}\mathbf{U} = -\frac{\mu}{r^2}\cos(\gamma)\mathbf{T} + \frac{\mu}{r^2}\sin(\gamma)\mathbf{N}$$

And so by substitution equation (7) becomes,

$$\mathbf{a} = \left[a_t - \frac{\mu}{r^2}\cos(\gamma) \right] \mathbf{T} + \frac{\mu}{r^2}\sin(\gamma)\mathbf{N} \ .$$

Hence,

$$\frac{dv}{dt} = a_t - \frac{\mu}{r^2}\cos(\gamma) \tag{8}$$

and

$$\kappa v^2 = \frac{\mu}{r^2}\sin(\gamma) \ . \tag{9}$$

Note that in polar coordinates the arc length can be stated as

$$(ds)^2 = (dr)^2 + (r d\theta)^2$$

where

$$r\frac{d\theta}{ds} = \sin(\gamma)$$

and

$$\frac{dr}{ds} = \cos(\gamma) \ .$$

So equation (8) becomes,

$$v\frac{dv}{dt} = a_t - \frac{\mu}{r^2}\frac{dr}{ds} \tag{10}$$

and equation (9) becomes,

$$\kappa v^2 = \frac{\mu}{r}\frac{d\theta}{ds} \ .$$

Then

$$v\frac{dv}{ds} = \frac{1}{2}\frac{dv^2}{ds} \ .$$

So the tangential acceleration a_t in equation (10) becomes

$$\frac{1}{2}\frac{dv^2}{ds} + \frac{\mu}{r^2}\frac{dr}{ds} = a_t \ .$$

Also by using polar coordinates to solve for κ we obtain the following,

$$\kappa = \frac{1}{v}\left(1 - \left(\frac{dr}{ds}\right)^2 - r\frac{d^2r}{ds^2}\right)\left(1 - \left(\frac{dr}{ds}\right)^2\right)^{-\frac{1}{2}}$$

which implies

$$\frac{v^2}{r}\left(1 - \left(\frac{dr}{ds}\right)^2 - r\frac{d^2r}{ds^2}\right)\left(1 - \left(\frac{dr}{ds}\right)^2\right)^{-\frac{1}{2}} = \frac{\mu}{r}\frac{d\theta}{ds} \ . \tag{11}$$

Note that

$$1 = \left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2$$

then

$$1 - \left(\frac{dr}{ds}\right)^2 = r^2\left(\frac{d\theta}{ds}\right)^2 .$$

Solving for $\frac{d\theta}{ds}$ we get,

$$\frac{d\theta}{ds} = \frac{1}{r}\sqrt{1 - \left(\frac{dr}{ds}\right)^2} .$$

Then the right-hand-side of equation (11) becomes

$$\frac{\mu}{r}\frac{d\theta}{ds} = \frac{\mu}{r^2}\sqrt{1 - \left(\frac{dr}{ds}\right)^2} .$$

Which can be written as,

$$\frac{\mu}{r}\frac{d\theta}{ds} = \frac{\mu}{r^2}\left(1 - \left(\frac{dr}{ds}\right)^2\right)\left(1 - \left(\frac{dr}{ds}\right)^2\right)^{-\frac{1}{2}} .$$

Hence equation (11) becomes

$$\frac{v^2}{r}\left(1 - \left(\frac{dr}{ds}\right)^2 - r\frac{d^2r}{ds^2}\right)\left(1 - \left(\frac{dr}{ds}\right)^2\right)^{-\frac{1}{2}} = \frac{\mu}{r^2}\left(1 - \left(\frac{dr}{ds}\right)^2\right)\left(1 - \left(\frac{dr}{ds}\right)^2\right)^{-\frac{1}{2}} .$$

And so by multiplying both sides by r and canceling out $\left(1 - \left(\frac{dr}{ds}\right)^2\right)^{-\frac{1}{2}}$ we now have,

$$v^2\left(1 - \left(\frac{dr}{ds}\right)^2\right) - rv^2\frac{d^2r}{ds^2} = \frac{\mu}{r}\left(1 - \left(\frac{dr}{ds}\right)^2\right) .$$

Thus,

$$rv^2\frac{d^2r}{ds^2} + \left(v^2 - \frac{\mu}{r}\right)\left(\left(\frac{dr}{ds}\right)^2 - 1\right) = 0 .$$

If we assume $\frac{d^2r}{ds^2} = 0$, this implies $v^2 = \frac{\mu}{r}$. So,

$$v^2 = 2sa_t + \mu\left(\frac{2}{r} - \frac{2}{r_o}\right) . \qquad (12)$$

Equation (12) is an integral for continuous thrust. In [1], an assumption allowed this to be used to approximate a solution for a continuous thrust orbit.

3 RESULTS

These are the results of research on the two body problem with a rocket.

3.1 Generalized Angular Momentum

To expand upon the results in section 1.2, we define the generalized angular momentum by

$$\mathbf{L} = \mathbf{r} \times \lambda \mathbf{v}$$

where λ is a function

$$\dot{\lambda} = \epsilon \alpha \lambda$$

$\mathbf{L}$ is constant, and so $\mathbf{r}$ must lie in a plane with normal $\mathbf{L}$. This is illustrated by the following theorem.

Theorem 3.1 *The generalized angular momentum* $(\mathbf{L})$ *is constant if and only if* $\dot{\lambda} = \epsilon \alpha \lambda$*.*

Proof: $(\Rightarrow)$ Suppose the generalized angular momentum $\mathbf{L}$ is constant. That is, suppose $\frac{d\mathbf{L}}{dt} = \mathbf{0}$.
The derivative of $\mathbf{L}$ is

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\left(\mathbf{r} \times \lambda \mathbf{v}\right)$$

using the product rule, this gives us

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \lambda \mathbf{v} + \mathbf{r} \times \frac{d}{dt}[\lambda \mathbf{v}]$$

and

$$\frac{d\mathbf{L}}{dt} = \underbrace{\mathbf{v} \times \lambda \mathbf{v}}_{= \mathbf{0}} + \mathbf{r} \times \underbrace{\left[\dot{\lambda}\mathbf{v} + \lambda\dot{\mathbf{v}}\right]}_{Product\ \ Rule} .$$

So we have,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \left[\dot{\lambda}\mathbf{v} + \lambda\dot{\mathbf{v}}\right]$$

which implies

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \dot{\lambda}\mathbf{v} + \mathbf{r} \times \lambda\dot{\mathbf{v}} .$$

Recall that

$$\ddot{\mathbf{r}} + \epsilon\alpha\dot{\mathbf{r}} + \frac{k}{r^3}\mathbf{r} = \mathbf{0} .$$

So

$$\ddot{\mathbf{r}} = -\epsilon\alpha\dot{\mathbf{r}} - \frac{k}{r^3}\mathbf{r} = \dot{\mathbf{v}} .$$

After substitution for $\dot{\mathbf{v}}$

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \dot{\lambda}\mathbf{v} + \mathbf{r} \times \lambda \left[-\epsilon\alpha\dot{\mathbf{r}} - \frac{k}{r^3}\mathbf{r}\right] .$$

Then

$$\frac{d\mathbf{L}}{dt} = \dot{\lambda}\left[\mathbf{r} \times \mathbf{v}\right] - \mathbf{r} \times \epsilon\alpha\lambda\mathbf{v} - \underbrace{\mathbf{r} \times \lambda\frac{k}{r^3}\mathbf{r}}_{= \mathbf{0}} .$$

Then

$$\frac{d\mathbf{L}}{dt} = \dot{\lambda}\left[\mathbf{r} \times \mathbf{v}\right] - \epsilon\alpha\lambda\left[\mathbf{r} \times \mathbf{v}\right] .$$

So we have

$$\frac{d\mathbf{L}}{dt} = \left[\dot{\lambda} - \epsilon\alpha\lambda\right](\mathbf{r} \times \mathbf{v}) = \mathbf{0} .$$

If $\mathbf{r} \times \mathbf{v} = \mathbf{0}$ we are done, else this implies that

$$\dot{\lambda} - \epsilon\alpha\lambda = 0 .$$

Hence,

$$\dot{\lambda} = \epsilon\alpha\lambda \ .$$

Thus, if the generalized angular momentum ($\mathbf{L}$) is constant then $\dot{\lambda} = \epsilon\alpha\lambda$.

($\Leftarrow$) Suppose $\dot{\lambda} = \epsilon\alpha\lambda$. We begin with,

$$\mathbf{L} = \mathbf{r} \times \lambda\mathbf{v}$$

and so the derivative of $\mathbf{L}$ is

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\left(\mathbf{r} \times \lambda\mathbf{v}\right)$$

using the product rule, this gives us

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \lambda\mathbf{v} + \mathbf{r} \times \frac{d}{dt}[\lambda\mathbf{v}]$$

and

$$\frac{d\mathbf{L}}{dt} = \underbrace{\mathbf{v} \times \lambda\mathbf{v}}_{=\ \mathbf{0}} + \mathbf{r} \times \underbrace{\left[\dot{\lambda}\mathbf{v} + \lambda\dot{\mathbf{v}}\right]}_{Product\ \ Rule} \ .$$

So we have,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \left[\dot{\lambda}\mathbf{v} + \lambda\dot{\mathbf{v}}\right]$$

which implies

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \dot{\lambda}\mathbf{v} + \mathbf{r} \times \lambda\dot{\mathbf{v}} \ .$$

Recall that

$$\ddot{\mathbf{r}} + \epsilon\alpha\dot{\mathbf{r}} + \frac{k}{r^3}\mathbf{r} = \mathbf{0} \ .$$

So

$$\ddot{\mathbf{r}} = -\epsilon\alpha\dot{\mathbf{r}} - \frac{k}{r^3}\mathbf{r} = \dot{\mathbf{v}} \ .$$

After substitution for $\dot{\mathbf{v}}$ and $\dot{\lambda}$

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \epsilon\alpha\lambda\mathbf{v} + \mathbf{r} \times \lambda \left[-\epsilon\alpha\dot{\mathbf{r}} - \frac{k}{r^3}\mathbf{r} \right] \ .$$

Then

$$\frac{d\mathbf{L}}{dt} = \epsilon\alpha\lambda \left[\mathbf{r} \times \mathbf{v} \right] - \mathbf{r} \times \epsilon\alpha\lambda\mathbf{v} - \underbrace{\mathbf{r} \times \lambda\frac{k}{r^3}\mathbf{r}}_{=\ \mathbf{0}} \ .$$

Then

$$\frac{d\mathbf{L}}{dt} = \epsilon\alpha\lambda \left[\mathbf{r} \times \mathbf{v} \right] - \epsilon\alpha\lambda \left[\mathbf{r} \times \mathbf{v} \right] \ .$$

So we have

$$\frac{d\mathbf{L}}{dt} = \underbrace{\left(\epsilon\alpha\lambda - \epsilon\alpha\lambda \right)}_{=\ 0} \left(\mathbf{r} \times \mathbf{v} \right) \ .$$

Hence,

$$\frac{d\mathbf{L}}{dt} = \mathbf{0} \ .$$

Thus, if $\dot{\lambda} = \epsilon\alpha\lambda$ then the generalized angular momentum ($\mathbf{L}$) is constant. Therefore, upon consideration of the last result, the generalized angular momentum ($\mathbf{L}$) is constant if and only if $\dot{\lambda} = \epsilon\alpha\lambda$. $\square$

For example, if α is constant, then

$$\dot{\lambda(t)} = \lambda(t)\frac{\dot{m}}{m}\alpha$$

which can be written as the separable ODE

$$\frac{d\lambda}{dt} = \lambda\frac{\dot{m}}{m}\alpha \ .$$

Then

$$\frac{d\lambda}{\lambda} = \frac{\dot{m}}{m}\alpha \ dt \ .$$

By solving this first order differential equation, we obtain a result of

$$\ln|\lambda| = \alpha \ln|m| + c \ .$$

Which is

$$\ln|\lambda| = \ln|m|^{\alpha} + c \ .$$

Application of the exponential yields

$$e^{\ln|\lambda|} = e^{\ln|m|^{\alpha}} e^{c} \ .$$

Hence

$$\lambda = c m^{\alpha} \ .$$

3.2 Perturbation Techniques

We define the generalized Lenz vector

$$\mathbf{F} = -\frac{k}{r}\mathbf{r} + \frac{1}{\lambda}(\mathbf{v} \times \mathbf{L})$$

which implies

$$\frac{d\mathbf{F}}{dt} = -2\epsilon\alpha\frac{1}{\lambda}(\mathbf{v} \times \mathbf{L}) \ .$$

Hence,

$$\frac{d\mathbf{F}}{dt} = -2\epsilon\alpha\left[\mathbf{F} + \frac{k}{r}\mathbf{r}\right] \ .$$

By distribution, we have

$$\underbrace{\frac{d\mathbf{F}}{dt} = -2\epsilon\alpha\mathbf{F}}_{homogeneous} - 2\epsilon\alpha\frac{k}{r}\mathbf{r} \ .$$

The homogeneous element is given by

$$\frac{d\mathbf{F}}{dt} = -2\epsilon\alpha\mathbf{F}$$

which implies a solution of

$$\mathbf{F} = \mathbf{c}e^{-2\int(\epsilon\alpha)dt}$$

where $\mathbf{c}$ is the constant of integration. Recall that $\frac{\dot{m}}{m} = \epsilon$ where we may assume $\epsilon \approx 0$, which implies little or no fuel loss.

3.3 The Main Result

The second Maple worksheet is based on the following theorem, which is a perturbation result since epsilon is close to 0.

Theorem 3.2 *Let $\gamma = \frac{-2\alpha\epsilon}{\lambda}$ be constant. Then, for some constant e, the solution to the rocket equation is*

$$r = \frac{L^2/\left(k\lambda\right)}{1 + e\cos(\theta)}$$

where

$$\lambda^2 = 1 - \int_0^\theta \frac{L^3\gamma/k^2}{1 + e\cos(u)} du \ .$$

Proof: We begin with the generalized Lenz vector

$$\mathbf{F} = -\frac{k}{r}\mathbf{r} + \frac{1}{\lambda}(\mathbf{v} \times \mathbf{L})$$

by differentiation we obtain

$$\begin{aligned}
\frac{d\mathbf{F}}{dt} &= \frac{k}{2r^3}\ \frac{d}{dt}\left(\mathbf{r}\cdot\mathbf{r}\right)\ \mathbf{r} - \frac{k}{r}\ \dot{\mathbf{r}} - \frac{\dot{\lambda}}{\lambda^2}\left(\dot{\mathbf{r}}\times\mathbf{L}\right) + \frac{1}{\lambda}\left(\ddot{\mathbf{r}}\times\mathbf{L}\right) \\
&= \frac{k}{2r^3}\ \left[2\left(\dot{\mathbf{r}}\cdot\mathbf{r}\right)\right]\ \mathbf{r} - \frac{k}{r^3}\left(\mathbf{r}\cdot\mathbf{r}\right)\dot{\mathbf{r}} - \frac{\dot{\lambda}}{\lambda^2}\left(\dot{\mathbf{r}}\times\mathbf{L}\right) \\
&\quad -\frac{1}{\lambda}\left(\alpha\epsilon\dot{\mathbf{r}}\times\mathbf{L}\right) - \frac{k}{\lambda r^3}\left(\mathbf{r}\times\mathbf{L}\right) \\
&= \frac{k}{r^3}\ \mathbf{r}\times\left(\mathbf{r}\times\dot{\mathbf{r}}\right) - \left(\frac{\dot{\lambda}}{\lambda^2} + \frac{\alpha\epsilon}{\lambda}\right)\left(\dot{\mathbf{r}}\times\mathbf{L}\right) - \frac{k}{r^3}\left[\mathbf{r}\times\left(\mathbf{r}\times\dot{\mathbf{r}}\right)\right] \\
&= -\left(\frac{\dot{\lambda}}{\lambda^2} + \frac{\alpha\epsilon}{\lambda}\right)\left(\dot{\mathbf{r}}\times\mathbf{L}\right)\ .
\end{aligned}$$

Note that

$$\frac{\alpha\epsilon}{\lambda} = \frac{\dot{\lambda}}{\lambda^2}$$

then

$$\frac{d\mathbf{F}}{dt} = -\frac{2\alpha\epsilon}{\lambda}\left(\dot{\mathbf{r}}\times\mathbf{L}\right)\ .$$

Hence

$$\frac{d\mathbf{F}}{dt} = \gamma\left(\dot{\mathbf{r}}\times\mathbf{L}\right)\ .$$

Integration yields

$$\mathbf{F} = \gamma\left(\mathbf{r}\times\mathbf{L}\right) + \mathbf{c_0}$$

where $\mathbf{c_0}$ is a constant of integration. Now we have

$$-\frac{k}{r}\mathbf{r} + \frac{1}{\lambda}(\dot{\mathbf{r}}\times\mathbf{L}) = \gamma\left(\mathbf{r}\times\mathbf{L}\right) + \mathbf{c_0}\ .$$

By applying the dot product with $\mathbf{r}$ we obtain

$$-\frac{k}{r}\mathbf{r}\cdot\mathbf{r} + \frac{1}{\lambda}(\dot{\mathbf{r}}\times\mathbf{L})\cdot\mathbf{r} = \gamma\left[\mathbf{r}\cdot\left(\mathbf{r}\times\mathbf{L}\right)\right] + \mathbf{c_0}\cdot\mathbf{r}$$

which implies

$$\begin{aligned}
-kr + \frac{1}{\lambda^2}(\mathbf{r} \times \dot{\mathbf{r}}) \cdot \mathbf{L} &= cr\cos(\theta) \\
-kr + \frac{L^2}{\lambda^2} &= cr\cos(\theta) \\
\frac{L^2}{\lambda^2} &= [k + c\cos(\theta)]\, r \ .
\end{aligned}$$

Hence, the solution to the rocket equation is

$$r = \frac{L^2/\lambda^2}{k + c\cos(\theta)} = \frac{L^2/\left(k\lambda^2\right)}{1 + e\cos(\theta)}$$

where e is a constant such that $e = \frac{c}{k}$. From this result it can be shown that

$$\lambda r^2 \frac{d\theta}{dt} = L$$

which implies

$$\frac{d\theta}{dt} = \frac{L}{\lambda r^2} \ .$$

Since $\gamma = \frac{-2\dot{\lambda}}{\lambda^2}$ we have

$$\lambda^2\gamma = -2\ \frac{d\lambda}{dt} = -2\ \frac{d\lambda}{d\theta}\ \frac{d\theta}{dt}$$

so

$$\lambda^2\gamma = -2\ \frac{d\lambda}{d\theta}\ \frac{L}{\lambda r^2}$$

then

$$\begin{aligned}
\frac{d\lambda}{d\theta} &= -\frac{\lambda^3\gamma r^2}{2L} \\
&= -\frac{\lambda^3\gamma}{2L}\left(\frac{L^2/\left(k\lambda^2\right)}{1 + e\cos(\theta)}\right)^2 \\
&= -\frac{\lambda^3\gamma L^4/k^2}{2\lambda^4 L\left(1 + e\cos(\theta)\right)^2} \ .
\end{aligned}$$

This implies

$$2\lambda \, d\lambda = -\frac{L^3\gamma/k^2}{(1+e\cos(\theta))^2} \, d\theta$$

and so integration of both sides from 0 to θ yields

$$(\lambda(\theta))^2 = (\lambda(0))^2 - \int_0^\theta \frac{L^3\gamma/k^2}{1+e\cos(\theta)} d\theta \; .$$

Since $\lambda(0) = 1$, and for computational purposes, the solution is

$$\lambda^2 = 1 - \int_0^\theta \frac{L^3\gamma/k^2}{1+e\cos(u)} du \; . \qquad \square$$

3.4 Results of the Maple Program

The Maple program that was created is an estimation for the trajectory of an elliptical orbit. However, the estimation is rather crude and we were only able to obtain a rough estimate. It follows the elliptical path close at first, unfortunately as time increased the solution we obtained deviated from the numeric solution. (See figure 3).

Using the same variables from the previous program, we implemented these into our calculations for the trajectory. We made a comparison to Maple's differential equation solver which finds a numerical solution using a Fehlberg fourth-fifth order Runge-Kutta method.

The Maple code for the estimation can be found in appendix 2.

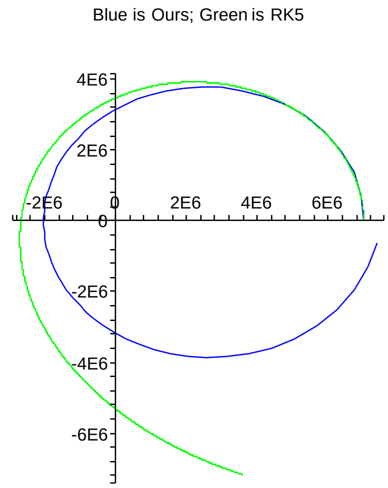

Figure 3: Our Estimation and Numeric Estimation of Rocket Trajectory.

4 CONCLUSION

The Lenz vector method for calculating trajectories is a good approximation technique. However, with every subsequent calculation there arose many hidden variables. One unexpected result was that the eccentricity e in

$$r = \frac{L^2/\left(k\lambda^2\right)}{1 + e\cos(\theta)}$$

does not appear to be constant. In the program, e was treated as a constant ($e = \frac{c}{k}$) and it is very likely to be a scalar function that may vary according to θ. From this point, there are many options for future research.

4.1 Future Research

A possible direction to take the research is to find a better approximation for the generalized Lenz vector $\mathbf{F}$. Let

$$\mathbf{F} = \alpha(t)\ \frac{-k}{r}\ \mathbf{r} + \beta(t)\left(\dot{\mathbf{r}} \times \mathbf{L}\right)$$

and find out what differential equations exist for $\alpha(t)$ and $\beta(t)$ that satisfy $\dot{\mathbf{F}} = \mathbf{0}$. The first differentiation alone gave us the result

$$\frac{d\mathbf{F}}{dt} = \dot{\alpha(t)}\ \frac{-k}{r}\ \mathbf{r} + \alpha(t)\left[\frac{d}{dt}\left(\frac{-k}{r}\right)\ \mathbf{r} + \frac{-k}{r}\ \dot{\mathbf{r}}\right] + \dot{\beta(t)}\left(\dot{\mathbf{r}} \times \mathbf{L}\right) + \beta(t)\ \frac{d}{dt}\left(\dot{\mathbf{r}} \times \mathbf{L}\right)\ .$$

The Maple differential equation solver was unable to compute a numeric solution to this equation. As we gain further knowledge about techniques in numerically solving differential equations, we may then apply those techniques to finding a solution to the above system.

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[4] Smith, Bert. Rocket Powered Flight for Keplerian Motion. Poster presented at the MAA undergraduate poster session on utilization of the rocket equation in an inverse square field. Joint meetings of the AMS–MAA. Washington D.C., January 2000.

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APPENDICES

Appendix A: Maple Code for Two Body Problem

This is the Maple code for the independent study midterm assignment.

```
> restart;

> with(linalg):

> with(LinearAlgebra):

> with(plots):

> with(plottools):
```

```
Initiating variables and constants.

> r0:=<7000000,0,0>:              # initial position

> v0:=<0,10000,0>:                # initial velocity

> M:=5.976*10^(24):               # mass of earth

> G:=6.672*10^(-11):              # gravity constant

> alpha:=0.05:                    # fuel loss proportion

> k:=M*G:                         # gravitation

> L:=norm(CrossProduct(r0, v0)):  # angular momentum

> h:=1/2*norm(v0)^2-k/norm(r0):   # hamiltonian energy

> e:=sqrt(1+2*h*((L^2)/(k^2))):   # eccentricity

> a:=L^2/(k*abs(e^2-1)):          # semi-major axis

> rho:=L^2/k:                     # easier for Maple
```

```
> c:=k/2:                         # easier for Maple
```

Plot the original ellipse and circle.

```
> r := theta -> (L^2/k)/(1+e*cos(theta)):
> pa1 := plot(r(theta),theta=0..2*Pi,coords=polar,color=black):
> x:=-1*a*e:
> pa2:= circle([x,0], a, color=red):
> display({pa1,pa2},scaling=constrained);
```

Incorporate these into a for loop.

```
> m:=0:
> for u from 0.0 by 0.1 to 6.28 do
>
>   if (h>0) then break end if:
>
>   t:=k^(-1/2)*a^(3/2)*(u-e*sin(u)):
>   r:=a*(1-e*cos(u)):
>   theta:=2*arctan(((((1+e))/(1-e))^(1/2))*tan(u/2)):
>
>   p1:=pointplot([r*cos(theta),r*sin(theta)],color=blue):
>   p2:=pointplot([a*cos(u)-a*e,a*sin(u)],color=blue):
>   p3:=line([-a*e,0],[a*cos(u)-a*e,a*sin(u)],color=red):
>   p4:=line([a*cos(u)-a*e,a*sin(u)],[a*cos(u)-a*e,r*sin(theta)],color=green):
```

```
>
>   ttle:=cat("time= ",convert(t,'string')):
>
>   m:=m+1:
>
>   plotpos[m]:=display({p1,p2,p3,p4},title=ttle):
>   tt[m]:=t:
> end do:
> tsync:=evalf( (tt[m]-tt[1])/50, 5 ):
> msync:=1:
> mcnt:=0:
> for ti from tt[1] to tt[m] by tsync do
>    while(tt[msync] < ti and msync < m) do
>       msync := msync+1:
>    end do:
>    mcnt:=mcnt+1:
>    ppp[mcnt]:=plotpos[msync]:
> end do:
> ani:=display(seq(ppp[i],i=1..mcnt),insequence=true):
> display(ani,pa1,pa2);
```

```
> restart:with(DEtools):with(linalg):with(LinearAlgebra):with(plots):
```

Define variables and constants.

```
> r0:=<7000000,0,0>:

> v0:=<0,5000,0>:

> epsilon:=0.001: # vary with gaamma

> alpha:=-0.1:

> L:=Norm(CrossProduct(r0, v0),2):

> M:=5.976*10^(24):

> G:=6.672*10^(-11):

> k:=M*G:

> h:=1/2*norm(v0)^2-k/norm(r0):

> e:=sqrt(1+2*h*((L^2)/(k^2))): #should be a function

>
```

Runge Kutta

```
> appsol:=dsolve({Diff(x(t),t)=u(t),Diff(y(t),t)=v(t),

>         Diff(u(t),t)=-epsilon*alpha*u(t)-k*x(t)/( (x(t)^2+y(t)^2)^(3/2) ),

>         Diff(v(t),t)=-epsilon*alpha*v(t)-k*y(t)/( (x(t)^2+y(t)^2)^(3/2) ),

>         u(0)=v0[1], v(0)=v0[2], x(0)=r0[1], y(0)=r0[2]},

>         {u(t),v(t),x(t),y(t)},method=rkf45,type=numeric):

> tt:=0:

> for i from 1 to 5000 do

>    xx[i]:=rhs(op(4,appsol(tt))):
```

```
>    yy[i]:=rhs(op(5,appsol(tt))):
>    tt:=tt+0.5;
> end do:
>
```

Estimate the trajectory using the generalized Lenz result.

```
> gaamma:=-0.2*alpha*epsilon: # vary with epsilon
> lkg:=evalf((-L^3/k^2*gaamma),15):
> i:=1:
> for theta from 0 to 6.28 by 0.1 do
>    lambda_thet:=1+lkg*evalf(Int((1-e*cos(u))^(-2),u=0..theta),15):
>    thet[i]:=theta;
>    r[i]:=(L^2/k)/(lambda_thet)/(1-e*cos(theta));
>    i:=i+1:
> end do:
> p1:=listplot([seq([r[j],thet[j]],j=1..(i-1))],coords=polar,color=blue):
> p2:=listplot([seq([xx[j],yy[j]],j=1..5000)],scaling=constrained,color=green):
> display(p1,p2,title="Blue is Ours; Green is RK5");
```

VITA

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